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THE LOGNORMAL RACE: A COGNITIVE-PROCESS MODEL OF CHOICE AND LATENCY WITH DESIRABLE PSYCHOMETRIC PROPERTIES

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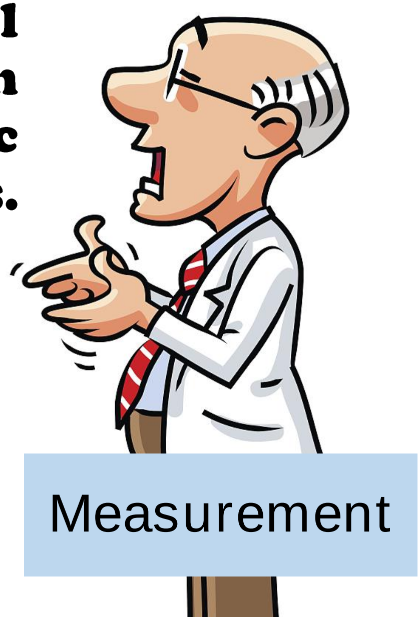
Introduction

- human performance data
  - measure individual abilities
  - measure the effects of covariates
  - assess latent structure

Cognitive

A best model should address the question of how these item responses are generated.

A best model is built on desirable statistical properties with measurement-theoretic assumptions.



Measurement

Introduction

- human performance data
  - measure individual abilities
  - measure the effects of covariates
  - assess latent structure

1. cognitive-process models
   - ✓ incorporate theoretical insights
   - ✓ to draw inferences about the processes and mental representations

finer process, but complex

2. psychometric models
   - ✓ retain desirable statistical properties
   - ✓ to incorporate covariates and understand variation in latent traits and abilities

statistically tractable, but coarse

➡ Purpose: develop a cognitive psychometrics model

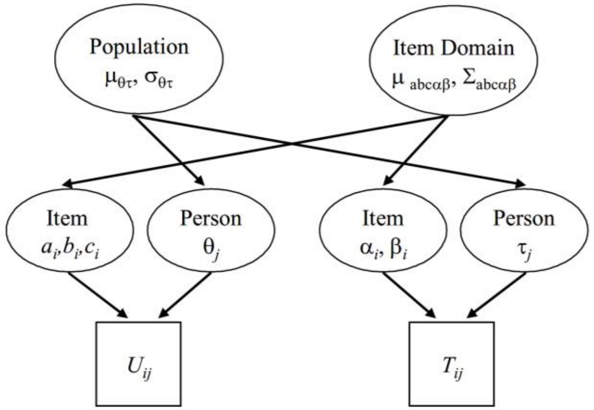
Modeling Response Time and Response Choice Jointly

- psychometric approach:

  1. place separate regression models on choice and RT
  2. allow RT / choice to be a covariate on the other one or share latent parameters

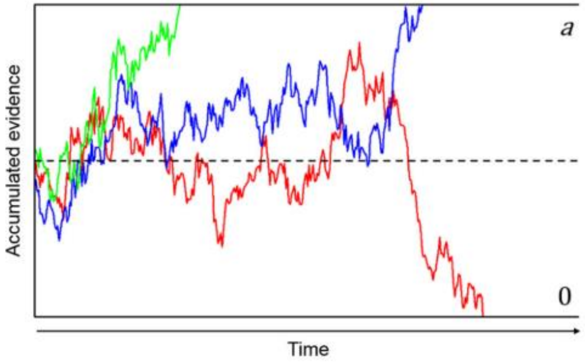
$$p_i(\theta_j) = [1 + \exp(\theta_j - \ln \tau_j - b_i)]^{-\pi_i}$$

$$\ln T_{ij} = \mu + \tau_j + \beta_i - \rho(a_i\theta_j - b_i) + \varepsilon_{ij}$$



- cognitive-process approach:
  - the diffusion model

$$f_{X,T}(x,t) = \frac{\pi\sigma^2}{\alpha^2}\exp\left(\frac{(\alpha x - z)(\theta + \beta)}{\sigma^2} - \frac{(\theta+\beta)^2}{2\sigma^2}(t - T_{er})\right)$$
$$\times \sum_{m=1}^{\infty} m \sin\left(\frac{\pi m(\alpha x - 2zx + z)}{\alpha}\right)\exp\left(-\frac{1}{2}\frac{\pi^2\sigma^2 m^2}{\alpha^2}(t - T_{er})\right)$$



Modeling Response Time and Response Choice Jointly

- adapting the diffusion model for psychometric settings
    - Tuerlinckx and De Boek (2005)

$$P_+ = P(X = 1) = \frac{e^{-2zv} - 1}{e^{-2av} - 1} = \frac{e^{av}}{1 + e^{av}}$$

    - van der Mass et al. (2011)

$$P_+ = \frac{e^{av}}{1 + e^{av}} = \frac{e^{\frac{a_k^p v_k^p}{a_j^i v_j^i}}}{1 + e^{\frac{a_k^p v_k^p}{a_j^i v_j^i}}}$$

    - Boehm et al. (2021)

$$\mathbb{P}(X = 1) = \frac{1}{1 + \exp\left(-\frac{v_{ip}}{cs_i}\right)}$$

➡

☐ test with an arbitrary number of choices

☐ the complexity of a multidimensional diffusion process

☐ the likelihood is not easily analyzed

☐ not clear whether more complex models may be placed on parameters

Specification of the Base Lognormal Race Model

- a simpler and more tractable cognitive process:
  - a race between accumulators

T = D / V

D $\sim B + U(0, A)$

$\sim e^{N(\mu_d, \sigma_d^2)}$

V $\sim N(v, sv)$

$\sim e^{N(\mu_v, \sigma_v^2)}$

  - the fastest accumulator indicates the finishing time

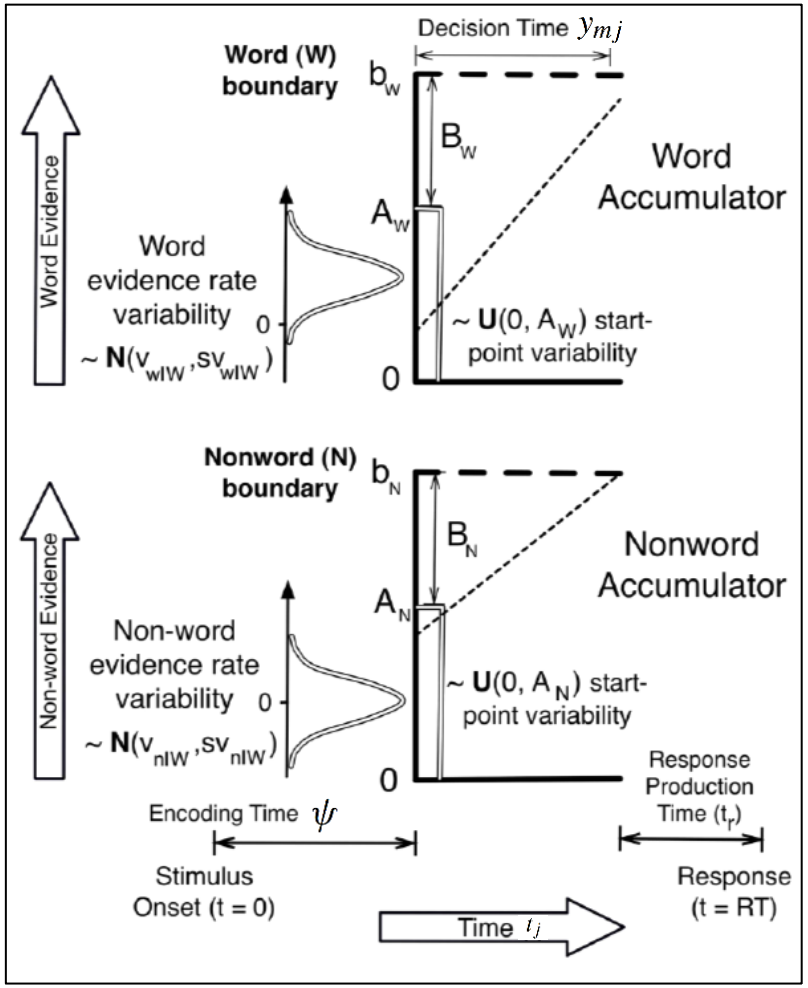
$$x_j = m \quad \Longleftrightarrow \quad y_{mj} = \min_i \boxed{(y_{ij})} \sim e^{N(\mu_d - \mu_v, \sigma_d^2 + \sigma_v^2)} = e^{N(\mu, \sigma^2)}$$

$$t_j = \psi + \min_i (y_{ij}) \qquad \overset{\text{ind}}{\sim} \text{Lognormal}\left(\mu_i, \sigma_i^2\right)$$

the joint density function:

$$f(m, t) = g\left(t - \psi; \mu_m, \sigma_m^2\right) \prod_{i \neq m} \left(1 - G\left(t - \psi; \mu_i, \sigma_i^2\right)\right)$$

Is “ *JUDGE* ” a real word?



Heathcote & Love, 2012 *Front Psychol*

Bayesian Analysis of the Base Model

- MCMC of the joint posterior distribution with Gibbs sampler

  1. prior specification

     $$\pi(\psi) \propto 1$$

     $$\sigma_i^2 \overset{\text{ind}}{\sim} \text{Inverse-Gamma}(a_i, b_i)$$

     $$\mu_i \overset{\text{ind}}{\sim} \text{Normal}(c_i, d_i)$$

  2. sampling from the conditional posterior distribution

     - conveniently expressed when conditioned on latent finishing times y_{ij}
     - let $z_{ij} = \log y_{ij}$ and then $\bar{z}_i = J^{-1} \sum_j z_{ij}$

➡ $$\mu_i \mid \cdots \sim \text{Normal}\big(v_i\big[J\bar{z}_i/\sigma_i^2 + c_i/d_i\big], v_i\big) \qquad v_i = \big(J/\sigma_i^2 + 1/d_i\big)^{-1}$$

$$\sigma_i^2 \mid \cdots \sim \text{Inverse-Gamma}\left(a_i + J/2, b_i + \sum_j (z_{ij} - \mu_i)^2/2\right)$$

$$f(\psi \mid \cdots) \propto \begin{cases} \prod_{ij} g(y_{ij} - \psi; \mu_i, \sigma_i^2), & \psi < \min_{ij}(y_{ij}) \\ 0, & \text{otherwise} \end{cases}$$

does not correspond to a known distribution
(Metropolis step with a symmetric normal random walk)

Bayesian Analysis of the Base Model

  - assume the i th accumulator finished first

$$(z_{ij} \mid x_j = i, \ldots) = \log(t_j - \psi)$$

$$(z_{ij} \mid x_j \neq i, \ldots) \sim \text{Truncated Normal}\big(\mu_i, \sigma_i^2, \log(t_j - \psi)\big)$$

- model comparison
  - the deviance information criterion [DIC]
    1. after each iteration in Gibbs sampling:

$$D^{(g)} = -2\ln\left(f(m,t) = g\big(t - \psi^{(g)}; \mu_m^{(g)}, \sigma_m^{2\,(g)}\big) \prod_{i \neq m} \big(1 - G\big(t - \psi^{(g)}; \mu_i^{(g)}, \sigma_i^{2\,(g)}\big)\big)\right)$$

$$\bar{D} = \frac{\sum_g D^{(g)}}{G}$$

    2. for the posterior mean:

$$D(\bar{\theta}) = -2\ln\left(f(m,t) = g\big(t - \psi; \mu_m, \sigma_m^2\big) \prod_{i \neq m} \big(1 - G\big(t - \psi; \mu_i, \sigma_i^2\big)\big)\right)$$

➡

$$p_D = \bar{D} - D(\bar{\theta})$$

$$\text{DIC} = \bar{D} + p_D$$

Application I: for Testing

- a Rasch-Like IRT Model
  - place backend model on the log finishing times z_{ij}

$$x_{jk} = m \quad \Longleftrightarrow \quad z_{mjk} = \min_i(z_{ijk})$$

$$t_{jk} = \psi_k + \exp\Big(\min_i(z_{ijk})\Big)$$

  - $z_{ijk} = \underbrace{\boxed{\alpha_{ij}}}_{\text{difficulty}} - \delta_{ij}\overbrace{\boxed{\beta_k}}^{\text{ability}} + \epsilon_{ijk}$ if i is the correct choice

$$z_{ijk} = \alpha_{ij} - \beta_k + \epsilon_{ijk}$$

otherwise

$$z_{ijk} = \alpha_{ij} + \epsilon_{ijk}$$

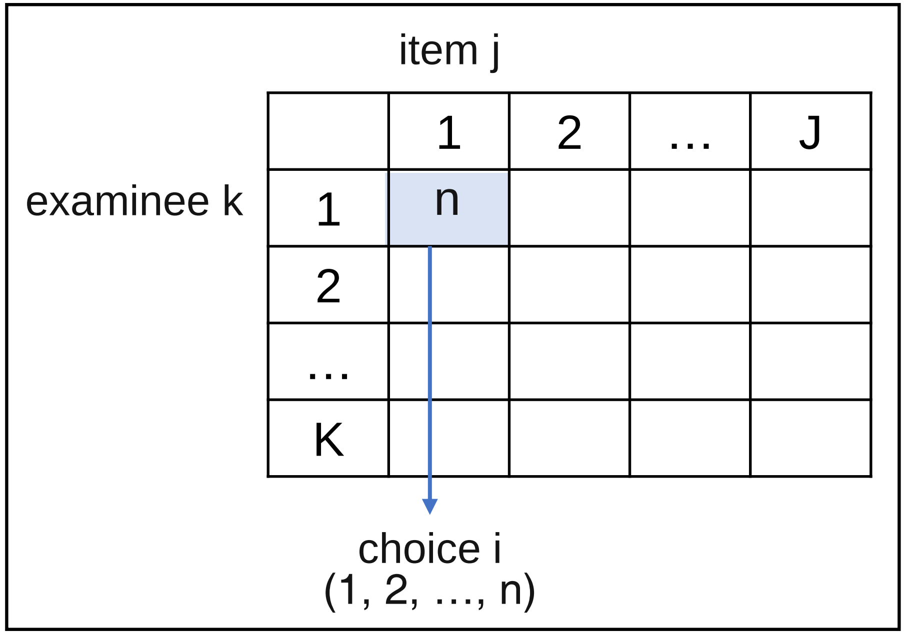
  - for the kth examinee:

$$z_k = (z_{11k}, \ldots, z_{n1k}, z_{12k}, \ldots, z_{nJk})'$$

across all examinees: $\boldsymbol{z} = (z_1', \ldots, z_K')'$

$$\boldsymbol{\alpha} = (\alpha_{11}, \ldots, \alpha_{n1}, \alpha_{1J}, \ldots, \alpha_{nJ})' \text{ and } \boldsymbol{\beta} = (\beta_1, \ldots, \beta_K)'$$

➡ $\boldsymbol{z} = \boldsymbol{X}_\alpha\boldsymbol{\alpha} - \boldsymbol{X}_\beta\boldsymbol{\beta} + \boldsymbol{\epsilon}$



item j

examinee k

	1	2	…	J
1	n			
2				
…				
K				

choice i
(1, 2, …, n)

Model development

- priors and posteriors

$$(z_{ij} \mid x_j = i, \ldots) = \log(t_j - \psi)$$

$$z_{ijk} = \boxed{\alpha_{ij}} - \delta_{ij}\boxed{\beta_k} + \boxed{\epsilon_{ijk}}$$

iid normally distributed zero-centered with variance σ^2

be modeled as random effect (with a hierarchical prior):

$$\boldsymbol{\beta} \mid \sigma_\beta^2 \sim \text{Normal}\left(\mathbf{0}, \sigma_\beta^2 \boldsymbol{I}\right)$$

$$\sigma_\beta^2 \sim \text{Inverse-Gamma}(a, b)$$

  - treat **α** and **β** as a single vector:

$$\boldsymbol{\theta} = (\boldsymbol{\alpha}', \boldsymbol{\beta}')'$$

$$\boldsymbol{z} = \boldsymbol{X\theta} + \boldsymbol{\epsilon} \quad \boldsymbol{X} = (\boldsymbol{X}_\alpha, \boldsymbol{X}_\beta)$$

➡ $\boldsymbol{\theta} \mid \cdots \sim \text{Normal}(\boldsymbol{\phi q}, \boldsymbol{\phi}) \quad \boldsymbol{\phi} = \left(\boldsymbol{X}'\boldsymbol{X}/\sigma^2 + \boldsymbol{B}\right)^{-1}$

$$\boldsymbol{q} = \boldsymbol{X}'\boldsymbol{z}/\sigma^2 + \boldsymbol{B\mu}_0$$

✓ prior precision: $\boldsymbol{B} = \text{diag}(\mathbf{1}_{nJ}/d, \mathbf{1}_K/\sigma_\beta^2)$

✓ prior mean: $\boldsymbol{\mu}_0 = (c\mathbf{1}_{nJ}, \mathbf{0}_K)'$

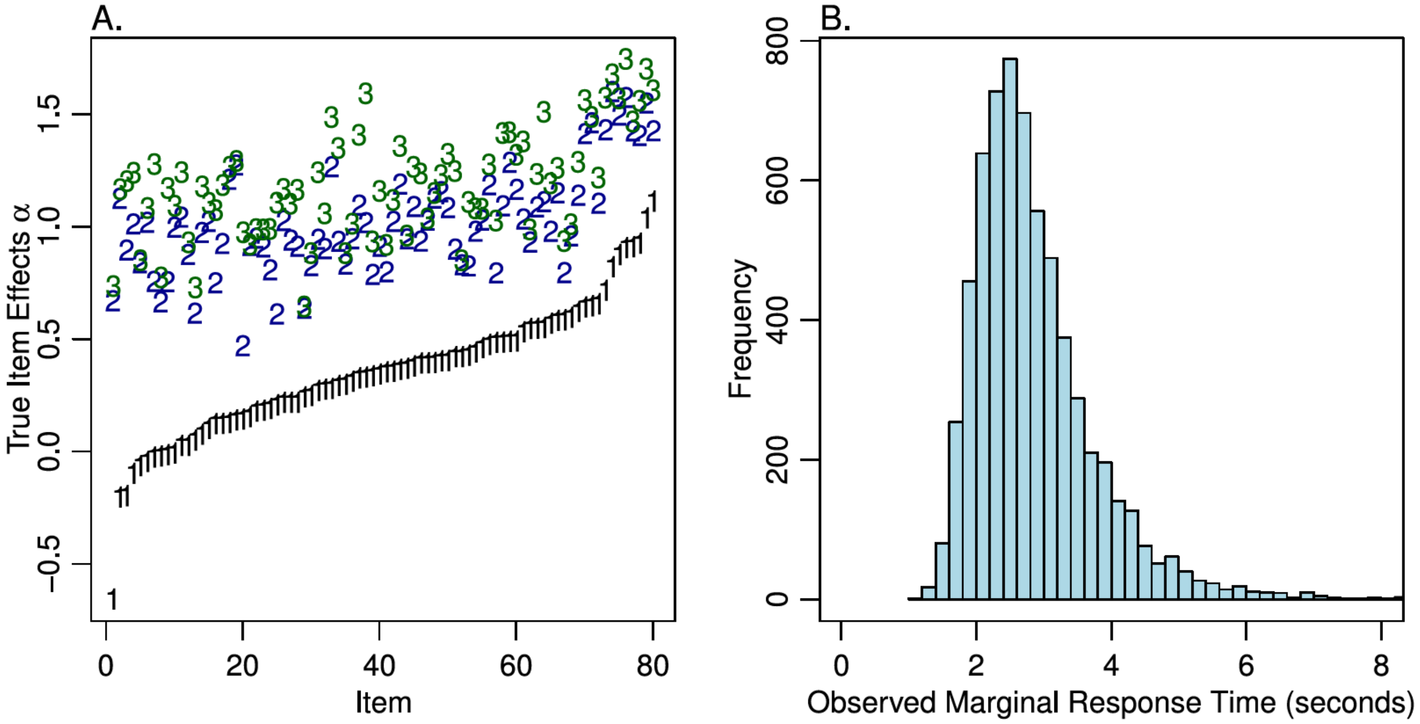
$$\sigma^2 \mid \cdots \sim \text{Inverse-Gamma}\left(a_1 + nJK/2, b_1 + (\boldsymbol{z} - \boldsymbol{X\theta})'(\boldsymbol{z} - \boldsymbol{X\theta})/2\right)$$

$$\sigma_\beta^2 \mid \cdots \sim \text{Inverse-Gamma}\left(a + K/2, b + \boldsymbol{\beta}'\boldsymbol{\beta}/2\right)$$

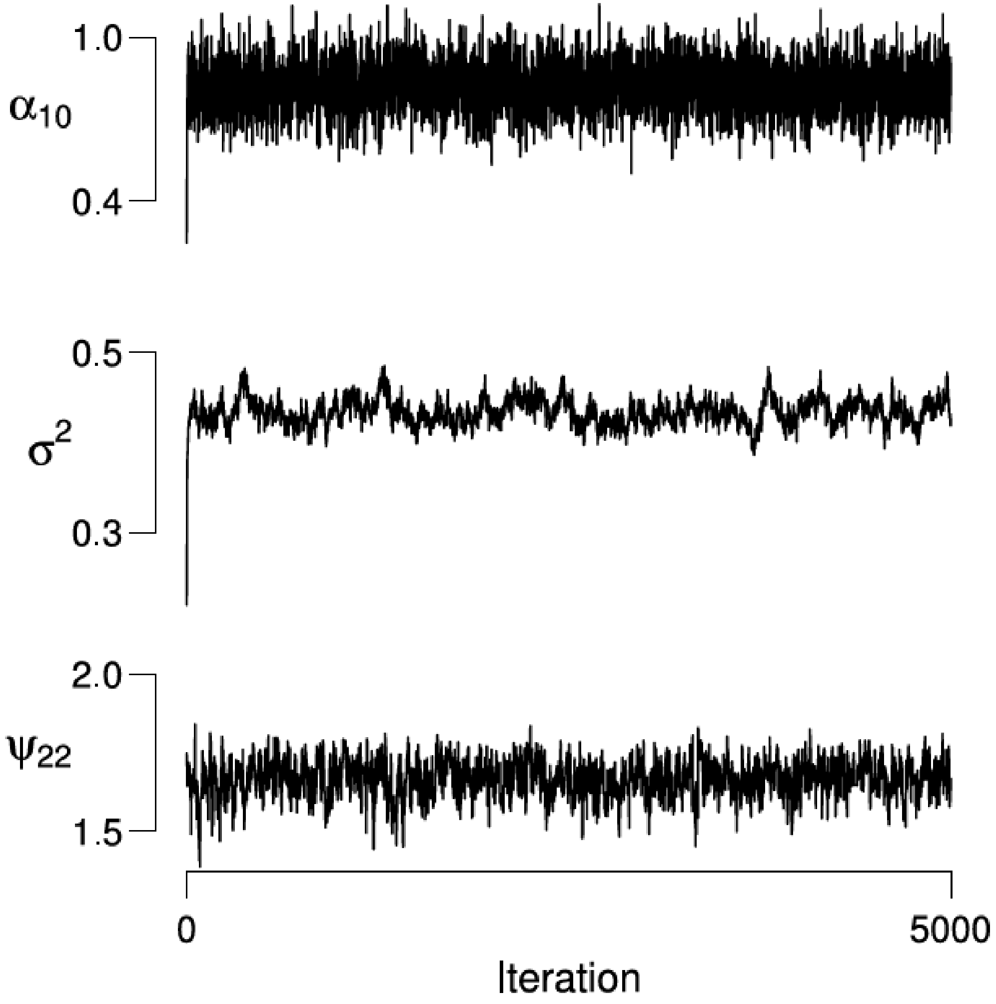
  - set c = 0 and d = 5
  - weakly informative: a = a1 = 1 and b = b1 = 0.1

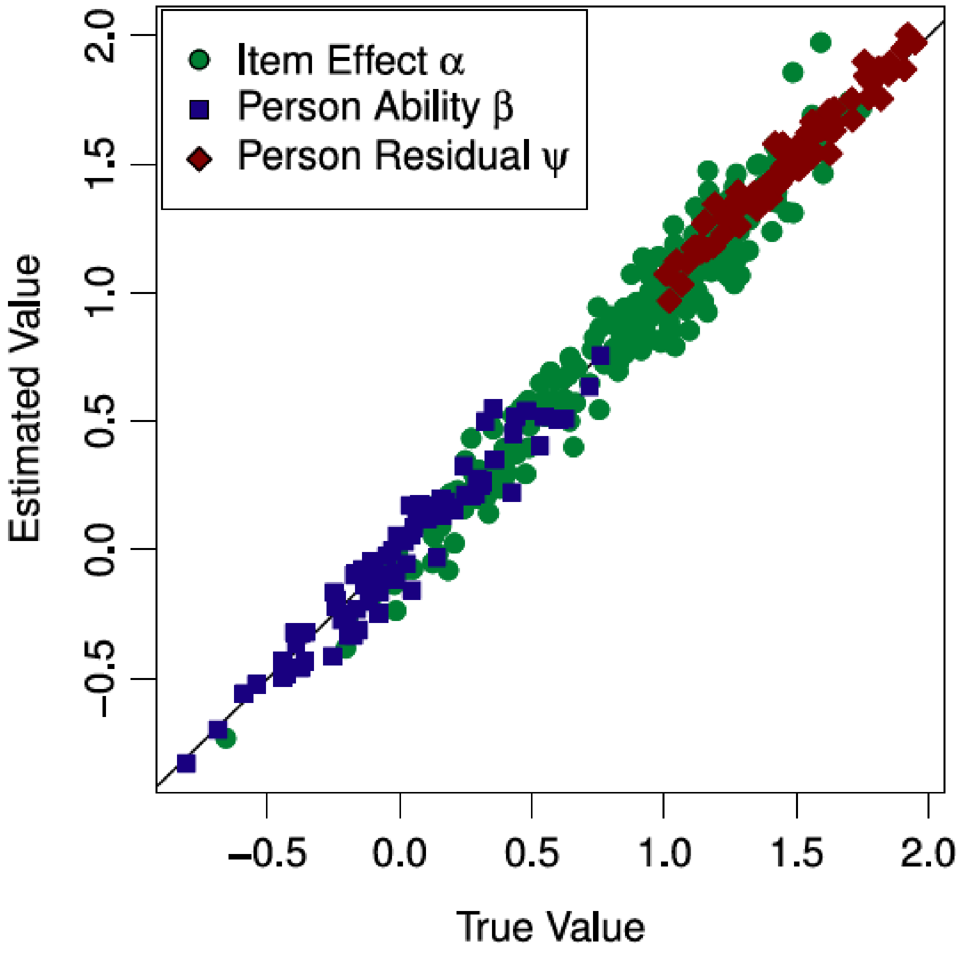
Whether the model can be analyzed efficiently

- simulated data set
  - 80 people
  - 80 items (three response options)
  - true shift values:
    $\psi_k \overset{\text{iid}}{\sim} \text{Unif}(1,2)$
  - MCMC:
    5 runs of 5,000 iterations
    a burn-in period of 500



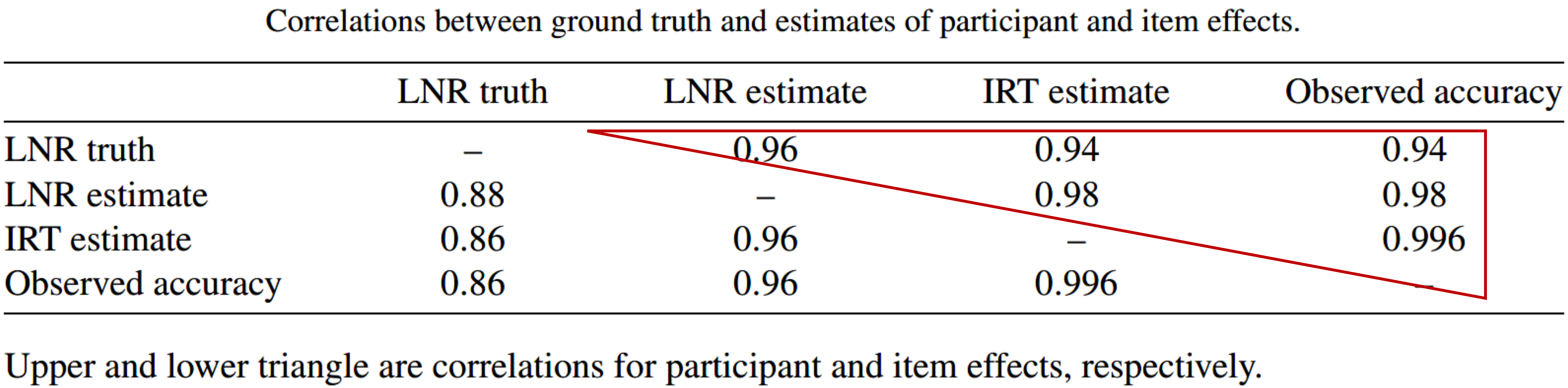
Results


1.0
α_{10}
0.4
0.5
σ^2
0.3
2.0
ψ_{22}
1.5
0
5000
Iteration


● Item Effect α
■ Person Ability β
◆ Person Residual ψ
Estimated Value
–0.5
0.0
0.5
1.0
1.5
2.0
True Value
–0.5
0.0
0.5
1.0
1.5
2.0

Results

- compared with the Rasch model



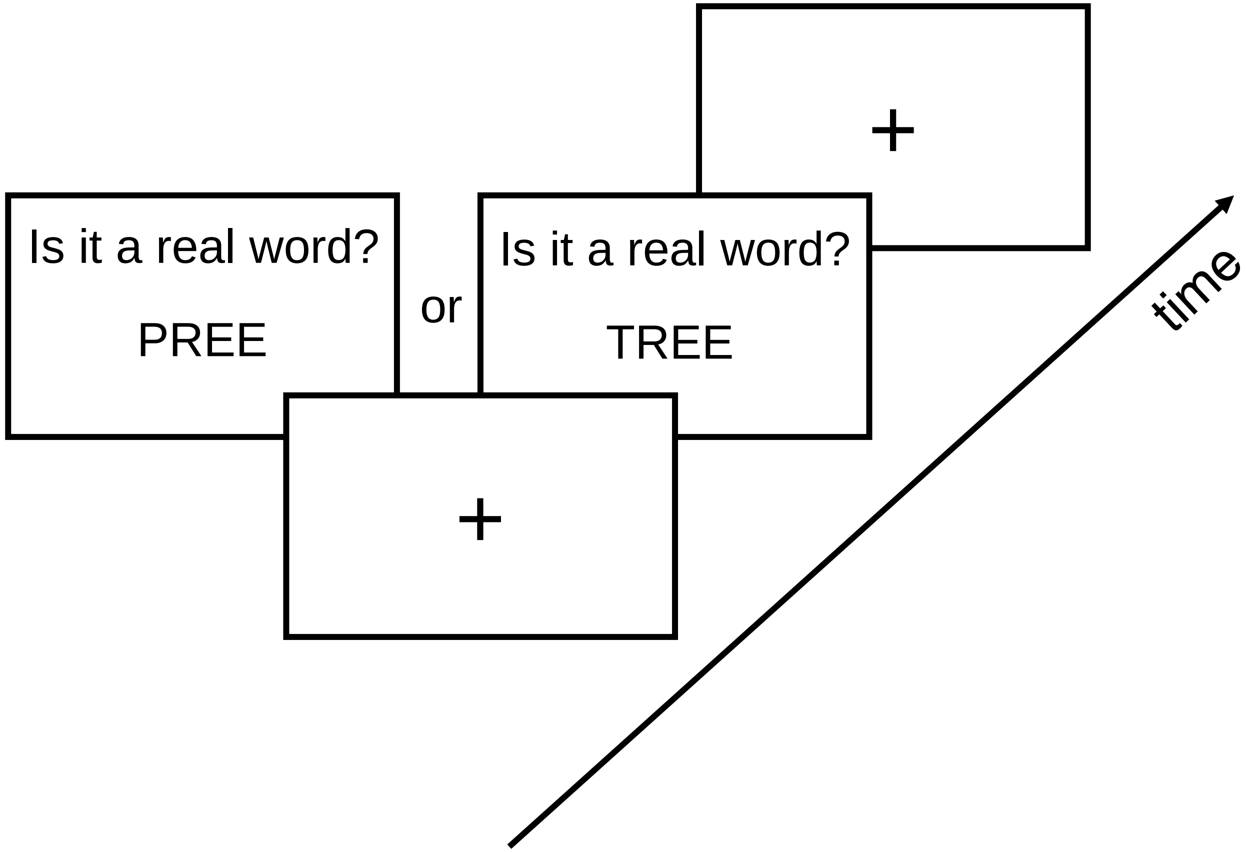
Correlations between ground truth and estimates of participant and item effects.

	LNR truth	LNR estimate	IRT estimate	Observed accuracy
LNR truth	–	0.96	0.94	0.94
LNR estimate	0.88	–	0.98	0.98
IRT estimate	0.86	0.96	–	0.996
Observed accuracy	0.86	0.96	0.996	–

Upper and lower triangle are correlations for participant and item effects, respectively.

Application II: for experimental psychology

- lexical decision task



Is it a real word?

PREE

or

Is it a real word?

TREE

+

+

time

- the data set:
  93 participants each performing 720 trials
- nonword:
  substitution conditions (5)
  e.g., TREE → PREE

  transposition conditions (7)

  e.g., JUDGE → JUGDE
- word:
  frequency of occurrence (3)
  e.g., CITY & AJAR

Model Specification

- autoregressive process: to model the trial order
  - $x_{jk} = 1$ (word) or 2 (nonword)

    z_{1k} and z_{2k} : latent log finishing times for the two accumulators (720 × 1)

    $\boldsymbol{\mu}_{1k}$ and $\boldsymbol{\mu}_{2k}$: participant-specific condition means (15 × 1)

    $\boldsymbol{X}_k$: maps trials into the 15 experimental conditions (720 × 15)

    ➡ residuals: $\boldsymbol{u}_{ik} = \boldsymbol{z}_{ik} - \boldsymbol{X}_k \boldsymbol{\mu}_{ik}$

  - first order autoregressive model

$$u_{ijk} = \rho u_{i,j-1,k} + \epsilon_{ijk}$$

$$\epsilon_{ijk} \overset{\text{iid}}{\sim} \text{Normal}(0, \sigma^2)$$

$$|\rho| < 1$$

$$u_{i,0,k} = 0$$

➡ noise terms:

$$\boldsymbol{\epsilon}_{ik} = \boldsymbol{A}_k \boldsymbol{u}_{ik}$$

(720 × 720)

$$\boldsymbol{A}_k = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -\rho & 1 \end{bmatrix}$$

Model Specification

- the joint expression

$$\boldsymbol{\epsilon}_{ik} = \boldsymbol{A}_k \boldsymbol{u}_{ik}$$

$$\boldsymbol{A}_k = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -\rho & 1 \end{bmatrix} \xrightarrow{\text{the inverse matrix}} \boldsymbol{L}_k = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \rho & 1 & \ddots & \ddots & & 0 \\ \rho^2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \rho^{J_k-2} & & \ddots & \ddots & 1 & 0 \\ \rho^{J_k-1} & \rho^{J_k-2} & \cdots & \rho^2 & \rho & 1 \end{bmatrix}$$

$$\boldsymbol{u}_{ik} = \boldsymbol{L}_k \boldsymbol{\epsilon}_{ik}$$

the joint distribution:

$$\boldsymbol{u}_{ik} \sim \text{Normal}(\mathbf{0}, \boldsymbol{\Sigma}_{\mathbf{k}})$$

$$\boldsymbol{\Sigma}_k = \sigma^2 \boldsymbol{L}_k \boldsymbol{L}_k{}'$$

the model on log finishing times:

$$z_{1k} \sim \text{Normal}(\boldsymbol{X}_k \boldsymbol{\mu}_k, \boxed{\boldsymbol{\Sigma}_k})$$

$$z_{2k} \sim \text{Normal}(\boldsymbol{X}_k \boldsymbol{\mu}_k, \boxed{\boldsymbol{\Sigma}_k})$$

nonzero, off-diagonal elements

Analysis

1. sampling the conditional posterior values of log finishing times
   - the winning accumulator:

     $$z_{\bar{i}jk} = \log(t_{jk} - \psi_k)$$

   - the losing accumulator:

     $$z_{ijk} | \cdots \sim \text{Truncated-Normal}\left(\mathrm{E}[z_{ijk}] + \rho u_{i,j-1,k}, \sigma^2, z_{\bar{i}jk}\right)$$

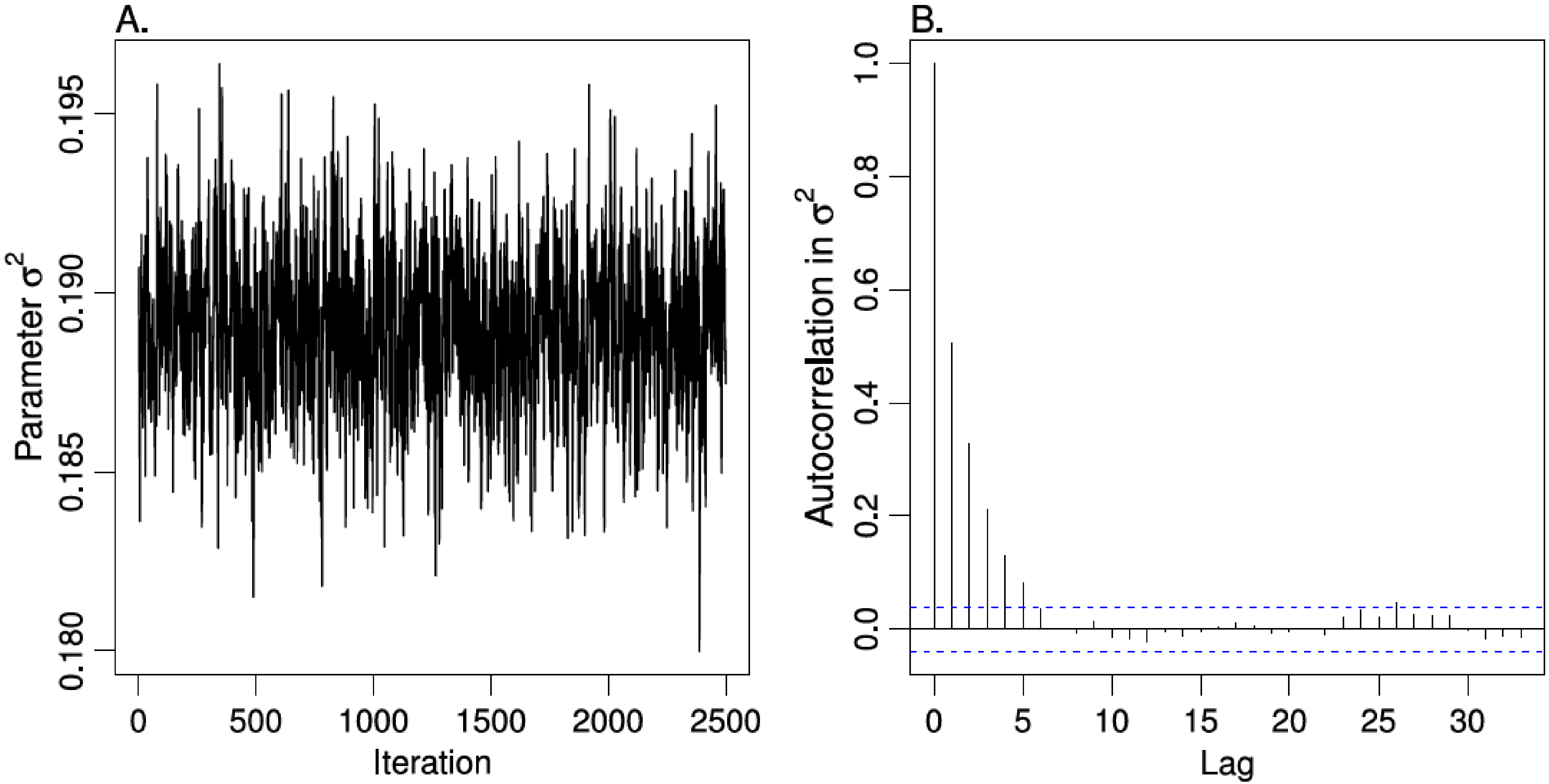
2. sampling the vectors of parameters and hyperparameters given the log finishing times
   - the autoregressive parameter:

     $$\rho \mid \cdots \sim \text{Truncated-Normal}_{(-1,1)}\left(m'/v', \sigma^2/v'\right)$$

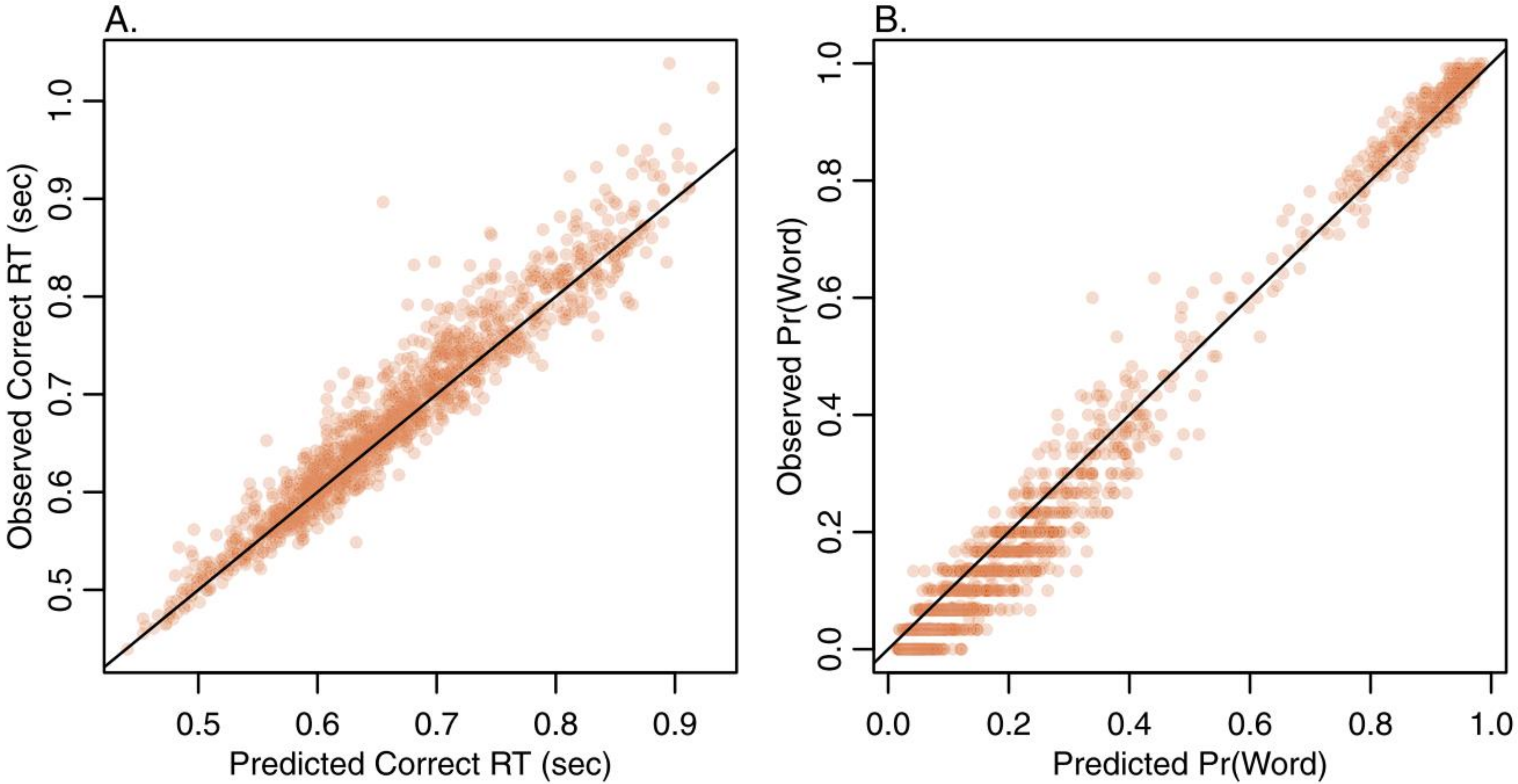
$$m' = \sum_{i=1}^{2} \sum_{k=1}^{K} \sum_{j=1}^{J_k - 1} u_{ijk} u_{i,j+1,k}$$

$$v' = \sum_{i=1}^{2} \sum_{k=1}^{K} \sum_{j=1}^{J_k - 1} u_{ijk}^2$$

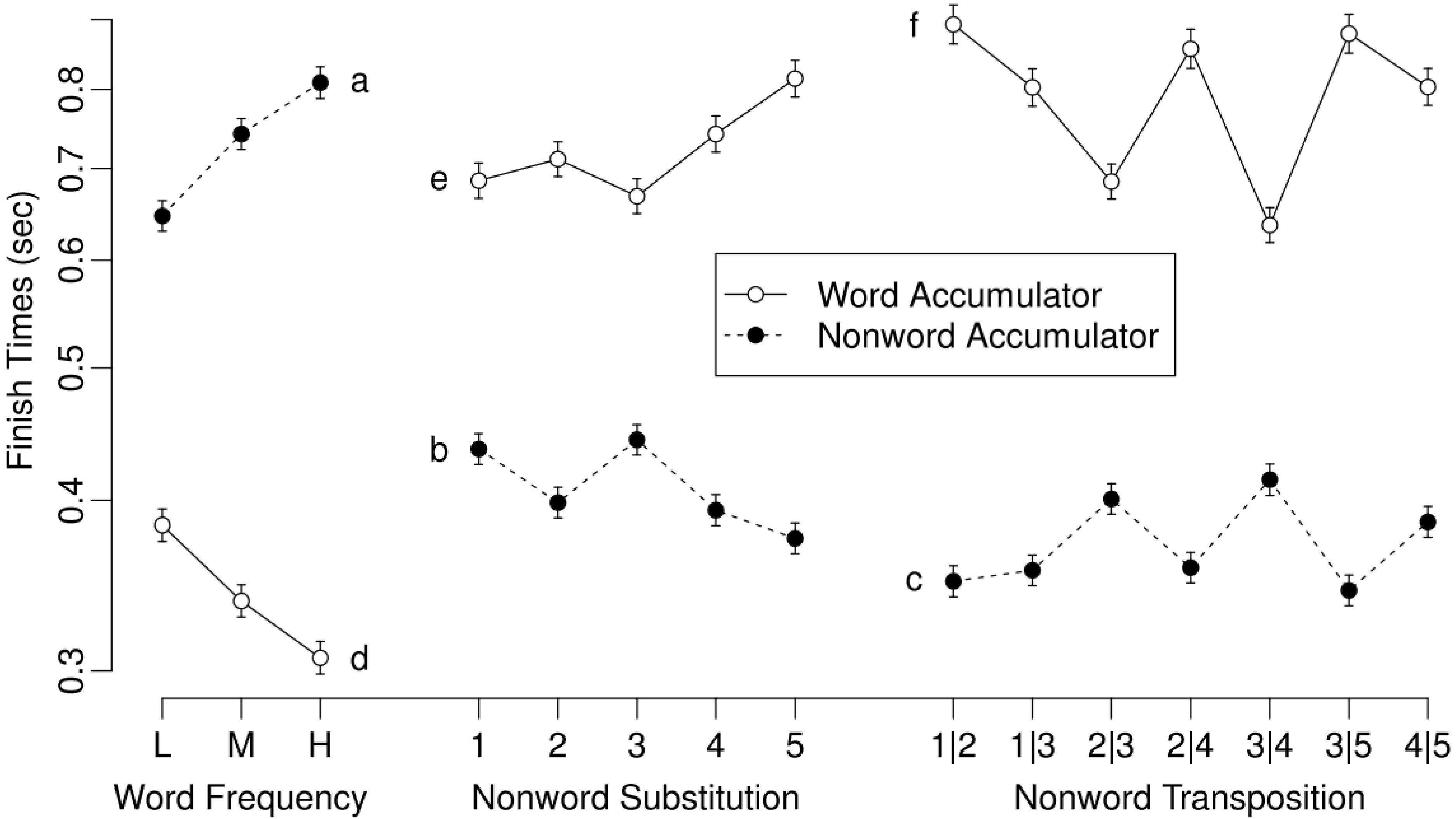
Results: mixing



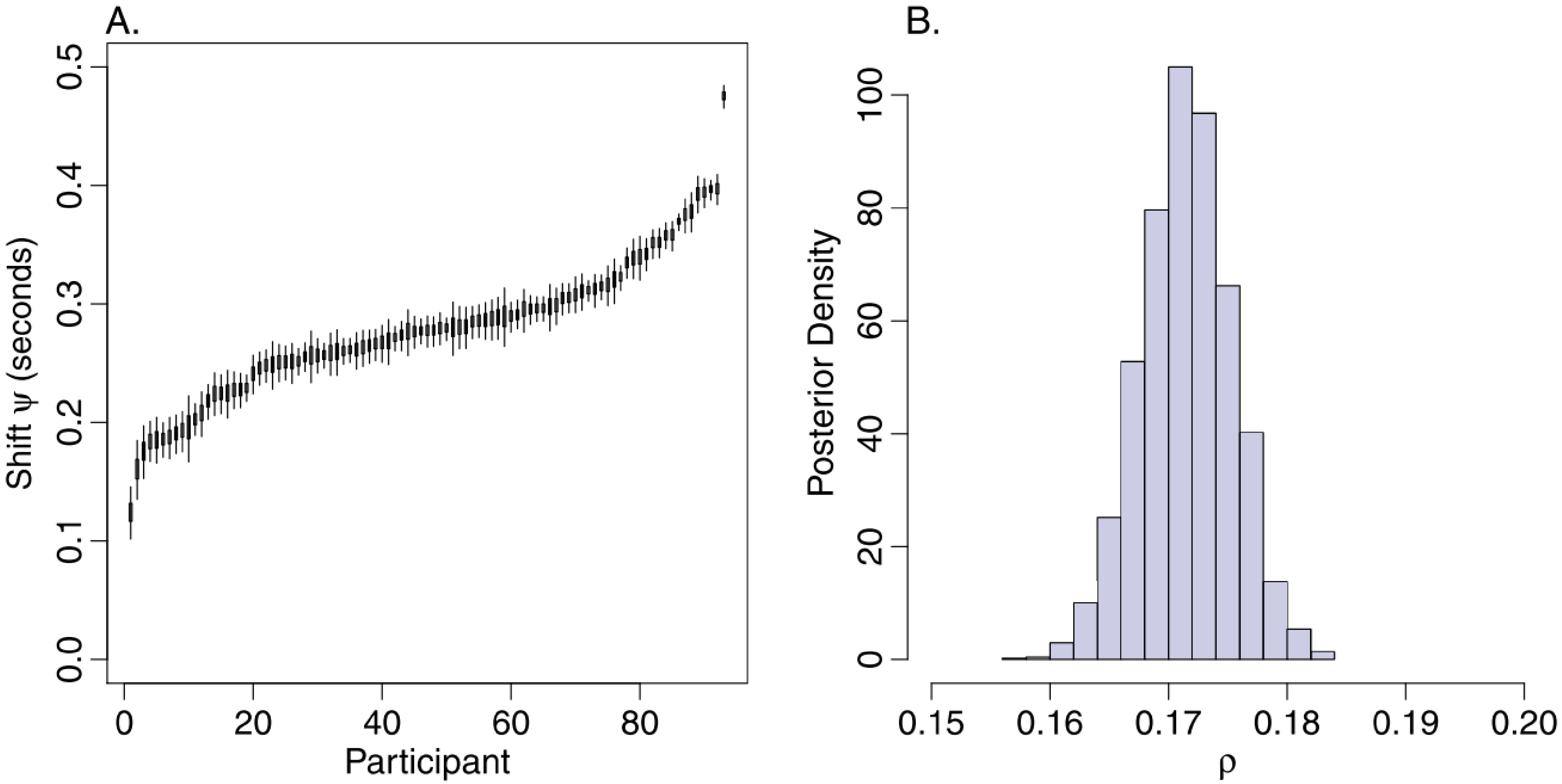
Results: model fit



Results: parameter estimates



Results: parameter estimates



DIC (without shift) = -68,879 > DIC (with shift) = -77,374

Discussion

- decision making process
  - arising from a race between competing evidence-accumulation

- straightforward to place sophisticated model components
  - IRT model & autoregressive model

Limitations

- bounds and accumulation rates cannot be disentangled
  - set bounds to constant 1: $y \sim \text{Lognormal}(\mu, \sigma^2)$

$$\mu = -\mu_\nu \text{ and } \sigma^2 = \sigma_\nu^2$$

  - specific parametric assumptions are needed to identify decision bounds
- highly accurate responses
  - the incorrect accumulators will largely reflect prior assumptions
- the additional development for a shift parameter
  - the inclusion of a Metropolis step & the impact on mixing under certain circumstances

THANKS FOR YOUR ATTENTION!

REPORTER

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